ACKNOWLEDGMENT

The authors wish to thank General Electric Company (Schenectady, NY) and the University of London for sponsorship during this research.

REFERENCES

[1] F. Y. Chang, "The generalized method of characteristics for waveform relaxation analysis of lossy coupled transmission lines," *IEEE Trans. Microwave Theory and Tech.*, vol. 37, no. 12, pp. 2028–2038, Dec. 1989.
[2] F. C. M. Lau and E. M. Deeley, "Improvements in the waveform relaxation method applied to transmission lines," submitted for publication.
[3] F. Y. Chang, "Waveform relaxation analysis of RLCG transmission lines," *IEEE Trans. Circuits Syst.*, vol. 37, no. 11, pp. 1394–1415, Nov. 1990.

A Useful Theorem for a Lossless Multiport Network

Xiaowei Shi, Changhong Liang, and Yiping Han

***Abstract*—A useful theorem is obtained for a lossless multiport network from the unitary condition of scattering matrix, and is proven to be equivalent to the unitary condition. Some illustrations are given to show how to apply the theorem to the analysis of the properties of a lossless n-port network.**

I. Introduction

In microwave engineering, many passive components can be taken as lossless. Therefore, analysis and synthesis of lossless networks are very important.

It is well known that the scattering matrix S of a lossless network meets the unitary condition

$$S^+S = I. \tag{1}$$

For a lossless two-port network, the following constrained conditions may result from (1) [1]:

$$|S_{11}| = |S_{22}|, \qquad |S_{12}| = |S_{21}| \tag{2}$$

$$\exp[j(\varphi_{11} + \varphi_{22})] = \exp\{j[(\varphi_{12} + \varphi_{21}) + \pi]\} = \det(S) \tag{3}$$

$$|S_{11}|^2 + |S_{21}|^2 = |S_{22}|^2 + |S_{12}|^2 = 1 \tag{4}$$

in which $S_{ik}(i, k = 1, 2)$ is the element of matrix $S, j = \sqrt{-1}, \varphi_{ik}$ is the phase of the element S_{ik}, $\det(S)$ means the determinant of matrix S.

For a lossless n-port network $(n > 2)$, things become complex. Multiplying S^+ with S and demanding the product-matrix equal unitary matrix, we can get n real equations and $n(n-1)/2$ complex equations. Those equations appear in a form different from (2) and (3). Also, they are not convenient to the analysis of the properties of a lossless n-port network.

Manuscript received June 15, 1993; revised June 27, 1994.
X. Shi and C. Liang are with the Microwave Laboratory, Department of Electromagnetic Engineering, Xidian University, Xian, P. R. China.
Y. Han is with the Department of Physics, Xidian University, Xian, P. R. China.
IEEE Log Number 9407448.

In 1991, Liang and Qiu [2] first found that the magnitude relation (2) may be generalized to a lossless n-port network. This paper shows that the phase relation (3) may also be generalized to a lossless n-port network. Furthermore, while the matrix S of a lossless n-port network $(n > 2)$ meets the generalized magnitude relation and generalized phase relation, any column (or row) of matrix S must be a complex unit vector (by using the term complex unit vector, we mean that the square sum of the magnitude of its elements equals 1), which is the generalized form of (4). That is to say, for a lossless n-port network $(n > 2)$, the generalized magnitude relation and phase relation are equivalent to the unitary condition (1). To be more important and meaningful, it is found that by using the generalized magnitude relation and generalized phase relation, the analysis of the properties of a lossless n-port network becomes much simpler. Three illustrations are given in this paper.

II. Two Theorems for Lossless Networks

Because the magnitude of the determinant of a scattering matrix must be 1 for lossless networks, in this paper, we will always let

$$\det(S) = \exp(j\varphi_D) \tag{5}$$

for a lossless network, where φ_D is the phase of the determinant of scattering matrix S.

Theorem 1: For a lossless n-port network, write its scattering matrix S in partitioned form

$$S = \begin{bmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{bmatrix}_{n\times n} \tag{6}$$

so that at least one of the two submatrix pairs $(S_{aa}, S_{bb}), (S_{ab}, S_{ba})$ is a square matrix pair. Let M_{ik}^c represent the cofactor of square submatrix $S_{ik}(i, k = a, b)$ in $\det(S)$; then we have

$$|\det(S_{ik})| = |M_{ik}^c| = |\det(S_{ki})| \qquad (i, k = a, b) \tag{7}$$

$$\arg[\det(S_{ik})] + \arg(M_{ik}^c) = \varphi_D \qquad (i, k = a, b) \tag{8}$$

or, equivalently,

$$[\det(S_{ik})]^* = \exp(-j\varphi_D)M_{ik}^c \qquad (i, k = a, b). \tag{9}$$

Remark: Applying Theorem 1 to a two-port network, we can get (2) from (7) and (3) from (8). Therefore, we call (7) and (8) the generalized magnitude relation and generalized phase relations, respectively. For convenience, in the following we will make use of (9) instead of (7) and (8).

Proof: The proof will be given only for the case that (S_{ab}, S_{ba}) is a square submatrix pair. A similar proof may be easily made for other cases.

Suppose S_{ab} is an $m\times m$ matrix. Then, S_{ba} is an $(n-m)\times(n-m)$ matrix, S_{aa} is an $m\times(n-m)$ matrix, S_{bb} is an $(n-m)\times m$ matrix. By applying the unitary condition (1) to partitioned matrix (6), we can get

$$\begin{aligned} S_{aa}^+S_{aa} + S_{ba}^+S_{ba} &= I_{n-m} \\ S_{ab}^+S_{ab} + S_{bb}^+S_{bb} &= I_m \\ S_{aa}^+S_{ab} + S_{ba}^+S_{bb} &= O_{(n-m)\times m} \end{aligned} \tag{10}$$

where I_m represents the $m \times m$ unit matrix, $O_{(n-m)\times m}$ represents an $(n-m) \times m$ zero matrix.

Let

$$U = \begin{bmatrix} O_{m\times(n-m)} & S_{ab} \\ I_{(n-m)} & O_{(n-m)\times m} \end{bmatrix}_{n\times n} \tag{11}$$

and multiply S^+ (from the right) by U

$$\begin{aligned} S^+U &= \begin{bmatrix} S_{aa}^+ & S_{ba}^+ \\ S_{ab}^+ & S_{bb}^+ \end{bmatrix} \begin{bmatrix} O_{m\times(n-m)} & S_{ab} \\ I_{(n-m)} & O_{(n-m)\times m} \end{bmatrix} \\ &= \begin{bmatrix} S_{ba}^+ & S_{aa}^+ S_{ab} \\ S_{bb}^+ & S_{ab}^+ S_{ab} \end{bmatrix}. \end{aligned} \tag{12}$$

Without changing the determinant value of the above block matrix, we may add the product of its first column with S_{bb} to its second column. Thus, we have

$$\begin{aligned} \det(S^+U) &= \det \begin{bmatrix} S_{ba}^+ & S_{aa}^+ S_{ab} + S_{ba}^+ S_{bb} \\ S_{bb}^+ & S_{ab}^+ S_{ab} + S_{bb}^+ S_{bb} \end{bmatrix} \\ &\overset{(10)}{=} \det \begin{bmatrix} S_{ba}^+ & O_{(n-m)\times m} \\ S_{bb}^+ & I_m \end{bmatrix} \\ &= [\det(S_{ba})]^*. \end{aligned} \tag{13}$$

On the other hand,

$$\det(S^+U) = [\det(S)]^* \det(U) = \exp(-j\varphi_D) M_{ba}^c. \tag{14}$$

Thus, we have proved (9) for $i = b, k = a$. Letting

$$U = \begin{bmatrix} O_{m\times(n-m)} & I_m \\ S_{ba} & O_{(n-m)\times m} \end{bmatrix}_{n\times n} \tag{15}$$

and making similar deduction, one may prove (9) for $i = a, k = b$.

The above proof clearly indicates that formulation (9) is an inevitable inference of the unitary condition (1). However, as we will prove, when n is greater than 2, the unitary condition (1) may also be deduced from (9). That is to say, formulation (9) is actually equivalent on the unitary condition (1) when n is greater than 2.

Theorem 2: Write the scattering matrix S of n-port network ($n > 2$) in partitioned form (6) so that at least one of the two submatrix pairs $(S_{aa}, S_{bb}), (S_{ab}, S_{ba})$ is a square matrix pair. If a network makes formulation (9) tenable for any square submatrix $S_{ik}(i, k = a, b)$ in every possible partitioned form, its scattering matrix must be unitary (i.e., the network is lossless).

Proof: Because formulation (9) is tenable for every partitioned form of S, we can choose a special partitioned form in which S_{aa} or S_{ab} is a submatrix of order 1 (i.e., an element of matrix S). To do so, in the following, $S_{ik}(i, k = 1, 2, \cdots, n)$ will be concerned as an element of matrix S, Δ_{ik} will be concerned as the cofactor of S_{ik} in $\det(S)$. By applying (9), we have

$$S_{ik}^* = \exp(-j\varphi_D)\Delta_{ik} \qquad (i, k = 1, 2, \cdots, n). \tag{16}$$

According to cofactor expansion of determinants [3], we can get

$$\sum_{i=1}^{n} S_{ik}^* S_{il} \overset{(16)}{=} \exp(-j\varphi_D) \sum_{i=1}^{n} \Delta_{ik} S_{il} \overset{(3)}{=} \exp(-j\varphi_D) \det(S)\, \delta_{kl}. \tag{17}$$

But the unitary condition (1) is equivalent to

$$\sum_{i=1}^{n} S_{ik}^* S_{il} = \delta_{kl}. \tag{18}$$

Therefore, Theorem 2 will be proven if we can prove $\det(S) = \exp(j\varphi_D)$.

Now, divide all possible cases into two types. 1) Every nondiagonal element $S_{ik}(i \neq k)$ equals zero. Thus, we can get $|S_{11}| = |S_{22}| = \cdots = |S_{nn}| = 1$ and $\det(S) = S_{11}S_{22}\cdots S_{nn} = \exp[j(\varphi_{11}+\varphi_{22}+\cdots+\varphi_{nn})]$ from (16). Apparently, this network is actually composed of n lossless one-port networks. 2) There is at least one nonzero, nondiagonal element. Noting the arbitrariness for port numbering, we can choose $S_{12} \neq 0$. Thus, according to (9), we have

$$S_{12} = \exp(j\varphi_D)\Delta_{12}^* = \exp(j\varphi_D)\sum_{i=2}^{n}(-S_{i1}D_{i1})^*. \tag{19}$$

While writing the last equality, we use the cofactor expansion again so that D_{i1} is the cofactor of S_{i1} in the determinant of $(-\Delta_{12})$ [the negative sign is due to the fact that Δ_{12} is a cofactor of S_{12} in the determinant $\det(S)$]. Now, considering the role of D_{i1} in $\det(S)$ and applying (9), we can get

$$D_{i1} = \exp(j\varphi_D)(S_{11}S_{i2} - S_{12}S_{i1})^*. \tag{20}$$

Inserting (20) into (19), we have

$$S_{12} = \sum_{i=2}^{n} S_{12}|S_{i1}|^2 - S_{11}\sum_{i=2}^{n} S_{i1}^* S_{i2} \overset{(17)}{=} S_{12}\sum_{i=1}^{n}|S_{i1}|^2. \tag{21}$$

Therefore,

$$1 \overset{(21)}{=} \sum_{i=1}^{n}|S_{i1}|^2 \overset{(17)}{=} \exp(-j\varphi_D)\det(S). \tag{22}$$

Thus, we have proven Theorem 2.

Actually, from the above proof, we can see that to identify a lossless network with formulation (9), one need not check all possible square submatrices; most relations will be automatically tenable while the remaining relations are tenable. If we note that the unitary condition (1) actually gives n^2 real equations for an n-port lossless network, the above conclusion is obvious.

III. Illustrations for Application of Theorem 1

Example 1: Consider a lossless three-port network. Being terminated from its port 3 with a load of reflection coefficient Γ, it is equivalent to a two-port network. The scattering matrix of the equivalent two-port network is [1]

$$S' = \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix} = \begin{bmatrix} \frac{S_{11}-\Delta_{22}\Gamma}{1-S_{33}\Gamma} & \frac{S_{12}+\Delta_{21}\Gamma}{1-S_{33}\Gamma} \\ \frac{S_{21}+\Delta_{12}\Gamma}{1-S_{33}\Gamma} & \frac{S_{22}-\Delta_{11}\Gamma}{1-S_{33}\Gamma} \end{bmatrix} \tag{23}$$

in which $S_{ik}(i, k = 1, 2)$ is the element of the scattering matrix of the original three-port network, Δ_{ik} is the cofactor of S_{ik}. If the load is a movable short (i.e., $|\Gamma| = 1$), then 1) if $|S_{12}| = |S_{21}|$, according to (9),

$$S_{12}/\Delta_{21} = \exp[j(\varphi_{12} + \varphi_{21} - \varphi_D)] = S_{21}/\Delta_{12}. \tag{24}$$

Therefore, when

$$\Gamma = \exp\{j[(\varphi_{12} + \varphi_{21}) + \pi - \varphi_D]\} \tag{25}$$

$S_{12}' = S_{21}' = 0$. That is, there will be no transmission of energy between port 1 and port 2 while the movable short takes an adequate position indicated by (25). 2) If $|S_{11}| = |S_{22}|$, according to (9),

$$S_{11}/\Delta_{22} = \exp[j(\varphi_{11} + \varphi_{22} - \varphi_D)] = S_{22}/\Delta_{11}. \tag{26}$$

Therefore, when

$$\Gamma = \exp\{j[(\varphi_{11} + \varphi_{22}) - \varphi_D]\} \tag{27}$$

$S_{11}' = S_{22}' = 0$. That is, there is a complete transmission of energy between port 1 and port 2 while the movable short takes an adequate position indicated by (27).

From the above, we see that lossless three-port networks have the following properties.

1) While $|S_{12}| = |S_{21}|$ for a lossless three-port network, a position can be found for a movable short terminating the port 3 of the network

for which there is no transmission of energy between the other two ports.

2) While $|S_{11}| = |S_{22}|$ for a lossless three-port network, a position can be found for a movable short terminating the port 3 of the network for which there is complete transmission of energy between the other two ports.

Apparently, the above properties are the generalization of the corresponding properties [1] of reciprocal three-port networks.

Example 2: Consider scattering bounds for a general lossless reciprocal three-port network. Butterweck [4] had pointed out that there are some bounds for the scattering parameters of a general reciprocal three-port network. By using Theorem 1, we can deal with this problem in a simpler manner.

According to (9), we have

$$S_{23}^* = \exp(-j\varphi_D)(S_{12}S_{13} - S_{11}S_{23}). \tag{28}$$

Therefore,

$$|S_{11}| = \left|\frac{S_{23}^*}{S_{23}}\exp(j\varphi_D) - \frac{S_{12}S_{13}}{S_{23}}\right| \geq \left|1 - \left|\frac{S_{12}S_{13}}{S_{23}}\right|\right|. \tag{29a}$$

Similarly, there are

$$|S_{22}| \geq \left|1 - \left|\frac{S_{12}S_{23}}{S_{13}}\right|\right| \tag{29b}$$

$$|S_{33}| \geq \left|1 - \left|\frac{S_{13}S_{23}}{S_{12}}\right|\right|. \tag{29c}$$

Noting $|S_{11}|^2 + |S_{22}|^2 + |S_{33}|^2 + 2(|S_{12}|^2 + |S_{13}|^2 + |S_{23}|^2) = 3$, we have

$$3 \geq \left[1 - \left(\left|\frac{S_{12}S_{13}}{S_{23}}\right| + \left|\frac{S_{12}S_{23}}{S_{13}}\right| + \left|\frac{S_{13}S_{23}}{S_{12}}\right|\right)\right]^2 + 2. \tag{30}$$

That is,

$$0 \leq \left|\frac{S_{12}S_{13}}{S_{23}}\right| + \left|\frac{S_{12}S_{23}}{S_{13}}\right| + \left|\frac{S_{13}S_{23}}{S_{12}}\right| \leq 2 \tag{31}$$

with (29a)–(29c) and (31), we can get further results (see [4]).

Actually, Theorem 1 is particularly useful for analyzing some special network (e.g., matched network and symmetrical network).

Example 3: Consider a matched lossless reciprocal five-port network (e.g., the so-called "star"); its scattering matrix is in the following form:

$$S = \begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{12} & 0 & S_{23} & S_{24} & S_{25} \\ S_{13} & S_{23} & 0 & S_{34} & S_{35} \\ S_{14} & S_{24} & S_{34} & 0 & S_{45} \\ S_{15} & S_{25} & S_{35} & S_{45} & 0 \end{bmatrix}. \tag{32}$$

Applying (9) to its 10 principal minors of order 2, we have

$$\begin{aligned}
-S_{12}^2 &= \exp(j\varphi_D)\cdot(2S_{34}S_{45}S_{35})^*, \\
-S_{13}^2 &= \exp(j\varphi_D)\cdot(2S_{24}S_{45}S_{25})^* \\
-S_{14}^2 &= \exp(j\varphi_D)\cdot(2S_{23}S_{35}S_{25})^*, \\
-S_{15}^2 &= \exp(j\varphi_D)\cdot(2S_{23}S_{34}S_{24})^* \\
-S_{23}^2 &= \exp(j\varphi_D)\cdot(2S_{14}S_{45}S_{15})^*, \\
-S_{24}^2 &= \exp(j\varphi_D)\cdot(2S_{13}S_{35}S_{15})^* \\
-S_{25}^2 &= \exp(j\varphi_D)\cdot(2S_{13}S_{34}S_{14})^*, \\
-S_{34}^2 &= \exp(j\varphi_D)\cdot(2S_{12}S_{25}S_{15})^* \\
-S_{35}^2 &= \exp(j\varphi_D)\cdot(2S_{12}S_{24}S_{14})^*, \\
-S_{45}^2 &= \exp(j\varphi_D)\cdot(2S_{12}S_{23}S_{13})^*.
\end{aligned} \tag{33}$$

Extracting the magnitude relations from (33), we have

$$\begin{aligned}
|S_{12}^2| &= |2S_{34}S_{45}S_{35}|, |S_{13}^2| = |2S_{24}S_{45}S_{25}| \\
|S_{14}^2| &= |2S_{23}S_{35}S_{25}|, |S_{15}^2| = |2S_{23}S_{34}S_{24}| \\
|S_{23}^2| &= |2S_{14}S_{45}S_{15}|, |S_{24}^2| = |2S_{13}S_{35}S_{15}| \\
|S_{25}^2| &= |2S_{13}S_{34}S_{14}|, |S_{34}^2| = |2S_{12}S_{25}S_{15}| \\
|S_{35}^2| &= |2S_{12}S_{24}S_{14}|, |S_{45}^2| = |2S_{12}S_{23}S_{13}|.
\end{aligned} \tag{34}$$

Noting the symmetry of (34), we can see that there are only two possible solutions:

$$|S_{12}| = \cdots = |S_{15}| = |S_{23}| = \cdots = |S_{45}| = 0 \tag{35}$$

or

$$|S_{12}| = \cdots = |S_{15}| = |S_{23}| = \cdots = |S_{45}| = 1/2. \tag{36}$$

Obviously, only solution (36) is meaningful. Also, it meets the unitary condition. As for the phase relations, only six independent relation may result

$$\exp[j(2\varphi_{12} + \varphi_{34} + \varphi_{45} + \varphi_{35})] = -\exp(j\varphi_D) \tag{37a}$$

$$\exp[j(2\varphi_{13} + \varphi_{24} + \varphi_{45} + \varphi_{25})] = -\exp(j\varphi_D) \tag{37b}$$

$$\exp[j(2\varphi_{14} + \varphi_{23} + \varphi_{35} + \varphi_{25})] = -\exp(j\varphi_D) \tag{37c}$$

$$\exp[j(2\varphi_{15} + \varphi_{23} + \varphi_{34} + \varphi_{24})] = -\exp(j\varphi_D) \tag{37d}$$

$$\exp[j(2\varphi_{23} + \varphi_{14} + \varphi_{45} + \varphi_{15})] = -\exp(j\varphi_D) \tag{37e}$$

$$\exp[j(2\varphi_{24} + \varphi_{13} + \varphi_{35} + \varphi_{15})] = -\exp(j\varphi_D). \tag{37f}$$

While a matched lossless five-port network is rotationally symmetrical, that is, $S_{12} = S_{23} = S_{34} = S_{45} = S_{15}$, $S_{13} = S_{35} = S_{25} = S_{24} = S_{14}$, there are

$$\varphi_{12} - \varphi_{13} = \pm 2\pi/3. \tag{38}$$

The results (36) and (38) were rigorously proven by Heiber [5], but the procedure here seems much simpler and clearer.

IV. Conclusion

Two theorems are proposed in this paper for a lossless network. The above work shows that by using Theorem 1, the analysis of a lossless network often becomes simpler. Besides, Theorem 2 states that Theorem 1 is actually equivalent to the unitary condition of scattering matrix. Therefore, Theorem 1 may also be used to predict some special lossless networks with particular properties.

References

[1] D. M. Kearns and R. W. Beatty, *Basic Theory of Waveguide Junction and Introductory Microwave Network Analysis.* Oxford: Pergamon, 1967, pp. 48, 102–103, 108.

[2] C. H. Liang and C. X. Qiu, "Several theorems for lossless networks," *Acta Electronica,* vol. 19, no. 3, pp. 101–102, 109, June 1991 (in Chinese).

[3] P. Lancaster and M. Tismenetsky, *The Theory of Matrices,* 2nd ed. New York: Academic, 1985, pp. 32–39.

[4] H. J. Butterweck, "Scattering bounds for the general lossless reciprocal three-port," *IEEE Trans. Circuit Theory,* vol. CT-13, pp. 290–293, Sept. 1966.

[5] A. L. Heiber and R. J. Vernon, "Matching consideration of lossless reciprocal 5-port waveguide junctions," *IEEE Trans. Microwave Theory Tech.,* vol. MTT-21, pp. 547–552, Aug. 1973.